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Analysis of The Competition Model of Two Populations Around The Orbit of The Equilibrium Point

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Abstract: The competition of two populations model that represented in an ordinary differential equations system. This model describes about the competition of two population in general that is consist of the interspesies competition and the intraspesies competition. In ecology, the population dynamic is closely related to population growth, equilibrium, and stability. Equilibrium is represented by a point called the equilibrium point or fixed point. By analyzing the stability around the fixed point, it can be seen the carrying capacity of a system, which mean the optimal number of individual that can be supported by the environment. According to the analysis of the model obtained 4 fixed points, three of them are unstable and the other else is stable. The orbit of the system around fixed point visualized using software. The behavior orbit around fixed point of the model will move away from the unstable fixed point and move closer to the stable fixed point..

Keywords: Equilibrium, Orbit, competition of two population model, and the carrying capacity of a model.

Introduction

Mathematics is one of the branches of science that can be applied in several other branches of science. Mathematics has an efficient symbolic language, beautiful orderly properties, and quantitative analytical abilities that will help produce mathematical models needed to understand problems in various branches of science and everyday life (Jumainisa, S., Darmawijoyo. & Hartono, 2018). Real phenomena in everyday life can be described using the language of mathematics known as mathematical modeling (S. Side, W. Sanusi, and N. K. Rustan, 2020) , (Anggriani et al., 2018). Mathematical modeling was first introduced by R. Rose in 1911, known as the Ross Model. This is in line with (G. A. Ngwa and W. S. Shu, 2000), which states that mathematical modeling was first introduced in 1911 by Ross, known as the Ross Model. Subsequently, MacDonald further developed the Ross model in 1957, known as the Ross-MacDonald model. Until now, mathematical modeling continues to be applied in various fields of science, social sciences, economics, engineering, and other fields (Asmaidi, 2021). Familiar natural phenomena that often occur in the lives of living things include predation, competition, and symbiosis. In biology, the study of these phenomena is called ecology.

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Ecology is a branch of biology that studies living things such as humans, animals and plants that live together and influence each other in their environment. The increase in population members in an ecosystem causes population density to increase so that individuals must compete to fulfill the same life needs. This competition is called competitive interaction. In individual animals, the needs of life that are often contested include food, water sources, shelter or nesting and mating partners. These life needs greatly affect the carrying capacity of an environment/system. Carrying capacity is the maximum number of individuals that can be supported by the environment. Competition is divided into two, namely intraspecies competition and interspecies competition. Intraspecies competition occurs between members of an animal population. While interspecies competition occurs between one species of animal population and another species of animal population. An example of animal competition is two populations of spiders, Lycosa, which compete with Enoplognatha for food resources. Currently, there are very few studies on the specific model of competition between Lycosa and Enoplognatha. Therefore, this article uses a general model of the competitive interaction of two populations.

In the research (Nurhamiyawan et al., 2018), the population dynamics at an undefined equilibrium point and when both populations coexist were analyzed, so that the population dynamics that occur and the possibility of both populations coexisting can be understood. The two-population competition model is a mathematical model that describes the competition among individuals within a single population and the competition between two populations to meet their living needs (Moneim & Khalil, 2015). The population development model of two competing populations is represented in a system of autonomous nonlinear differential equations (D. K. Arrowsmith, n.d.).

Competition model is also known as the Lotka-Volterra model. Several studies have discussed the Lotka-Volterra model, including Stability Analysis and Limit Cycle in the Gause Type Predator-Prey Model. The results of this research indicate that a stable equilibrium point occurs when the number of individuals in the predator and prey populations is non-zero, and the type of stability meets the stability criteria. Meanwhile, the Gause type model will have a limit cycle if the acceleration of prey population growth is not equal to the deceleration (negative acceleration) of the predator population, or vice versa (Rosyid, 2012). Another research is the Population Stability of the Three-Species Lotka-Volterra Model with Equilibrium Points. The result of this research is the stability at the equilibrium point and the coordinate plane through numerical analysis. Next, several numerical examples are provided, which show that the solutions obtained using parameter values give an overview of the development of the three species (Monica, Ritanica, 2014). The next research is the Bifurcation Analysis on the Predator-Prey Mathematical Model with Two Predators. The results of this study show that the bifurcation occurring in this model can be seen from the changes in the number of equilibrium points and the changes in the stability of the equilibrium points. However, the name of the bifurcation cannot be determined due to the behavior of this model being different from the behavior of systems that generally undergo bifurcation (Listyana, 2016). a competition is the interaction between individuals within a population and the interaction between competing populations vying for the same life necessities such as food, shelter, etc., with one species not preying on another species, so the interaction between goats and cows grazing in the same area or tigers and lions hunting the same prey can be formulated in a model of competition between two populations (Clarke, 1954).

The mathematical model of competition between two populations is described in terms of a system of autonomous nonlinear differential equations. The model includes both intraspecies and interspecies competition. By knowing the behavior of the orbit around the fixed point, we can know the stability around the fixed point of the model. The stability around the fixed point is closely related to the capacity of the two populations. If around the fixed point is stable, then the fixed point is a point that represents the capacity of two populations in the system. Therefore, in this article, we analyze the fixed point and the type of stability around the fixed point, then observe the behavior of the orbit around the fixed point that is stable and the fixed point that is unstable.

Method

Type of Research

This type of research is a literature review method (literature study), namely all the information needed is obtained by reading and understanding the references related to the research. In this research, the orbital behaviour of the two-population competition model will be analysed.

Research Steps

The steps that will be taken in this research are as follows :

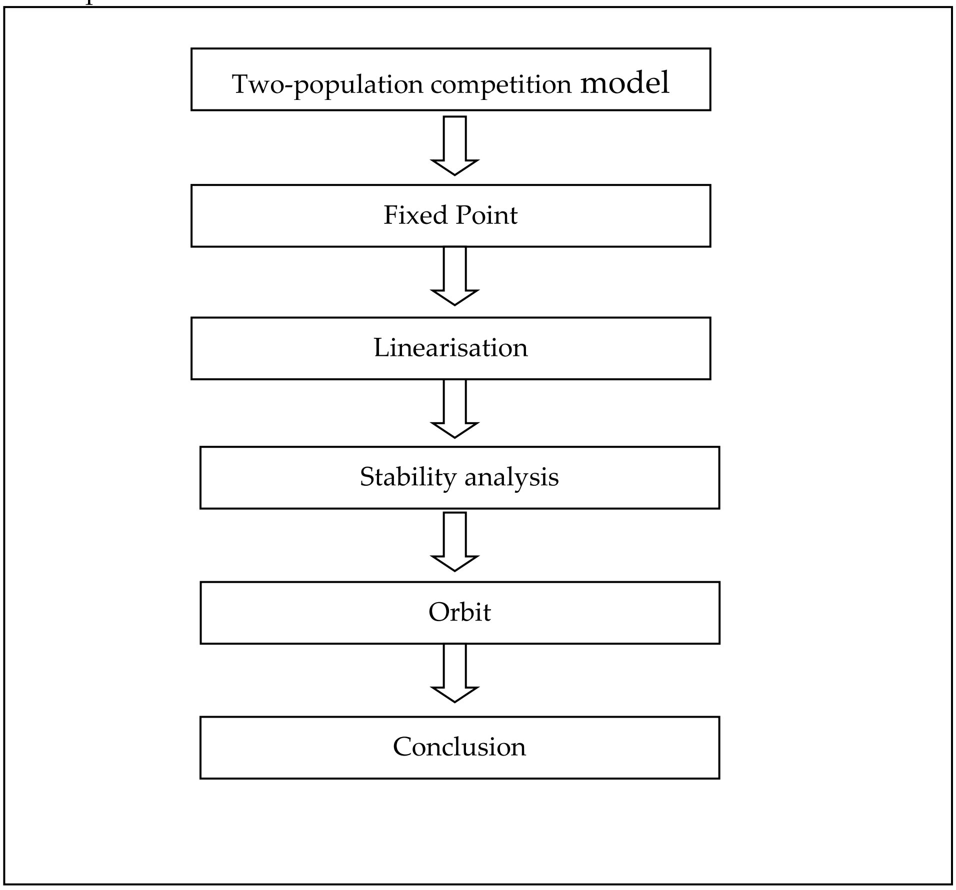

Figure 1. Flowchart of the research steps

The following is an explanation of the flowchart :

1. Fixed point of the two-population competition model

Analysing the fixed point is the first step in this article. The stability around the fixed point determines the capacity of the two-population competition model.

2. Linearisation

From the fixed point obtained, linearisation is carried out using the Jacobian matrix. Jacobian matrix is used to determine the eigenvalue (λ).

3. Stability analysis

The criteria for the type of stability around each fixed point can be seen from the eigenvalues obtained.

4. Orbit

The behaviour of the orbit around the fixed point is the trajectory of the movement of solutions $x= x(t)$ and $y=y(t)$ for $t \rightarrow \infty$ in the plane xy. Visualisation of the orbit (trajectory) of the two-population competition model using Matlab 15.2 software.

5. Conclusion

In this case the conclusions obtained are the fixed point, type and stability around the fixed point, as well as the behaviour of the orbit around the fixed point.

Result and Discussion

The two-population competition model is one of the models applied in the field of ecology. In ecology, there are several interactions, one of which is competition. Competition causes two populations that have ecological niches (place of life and function) very similar to each other cannot live together (coexist) in the same habitat continuously. Ecological niches involving resources, especially those vital for growth and reproduction, must be different in order to coexist in the same habitat. However, in reality, such niche distinction or separation may occur imperfectly, resulting in partial co-occurrence. Competition allows for changes in the number of individuals in a population and can lead to extinction. Therefore, to keep the two populations in the system from becoming extinct, it is necessary to know the capacity of the two populations in the system. In this study, the fixed point, stability around the fixed point, and orbital behaviour around the fixed point of the model are analysed. In ecology, the analysis of stability around the fixed point of the model of competition model is useful to know the capacity of the two populations in the system.

The two-population competition model is a mathematical model that model that describes the competition between individuals in one population or the competition between two populations for populations to get their needs. The two-population competition model is a system of autonomous nonlinear ordinary differential equations. Here is the model of competition between two populations:

$$\left.\begin{aligned} \frac{dN}{dt} &= N(a_1 - b_{11}N - b_{12}M) \\ \frac{dM}{dt} &= M(a_2 - b_{21}N - b_{22}M) \end{aligned}\right\}$$

with :

N : Number of first population
M : Number of second population
a_1 : Birth rate of the first population
a_2 : Birth rate of the second population
b_{11} : Mortality rate of the first population due to individual competition within the population
b_{21} : Mortality rate of the first population due to competition with the second population
b_{12} : Mortality rate of the second population due to competition with the first population
b_{22} : Mortality rate of the second population due to individual competition within the population

All parameters have positive values.

From the analysis, four fixed points were obtained: $E_1(0,0)$, $E_2\left(0,\frac{a_2}{b_{22}}\right)$, $E_3\left(\frac{(a_1b_{22}-a_2b_{12})}{(b_{11}b_{22}-b_{12}b_{21})},\frac{(a_2b_{11}-a_1b_{21})}{(b_{11}b_{22}-b_{12}b_{21})}\right)$, dan $E_4\left(\frac{a_1}{b_{11}},0\right)$. The fixed point E_1 represents the condition when the first population (N_0) and the second population (M_0) are both extinct. The fixed point E_2 represents the condition where the first population (N_0) is extinct and the second population (M_0) is alive. The fixed point E_3 represents the condition where both the first population (N_0) and the second population (M_0) are alive, meaning they coexist. Meanwhile, the fixed point E_4 represents the condition when the first population (N_0) is alive and the second population (M_0) is extinct. To determine the stability around the fixed point, the linearization of the two-population competition model must first be carried out. The Jacobian matrix of the system of differential equations for the competition of two populations is as follows:

$$J(N_0,M_0) = \begin{bmatrix} \frac{\partial N(t)}{\partial N} & \frac{\partial N(t)}{\partial M} \\ \frac{\partial M(t)}{\partial N} & \frac{\partial M(t)}{\partial M} \end{bmatrix}$$

$$J(N_0,M_0) = \begin{bmatrix} a_1 - 2b_{11}N - b_{12}M & -b_{12}N \\ -b_{21}M & a_2 - b_{21}N - 2b_{22}M \end{bmatrix} \quad (2)$$

The eigenvalues of the matrix $J(N_0, M_0)$ in equation (2) can be obtained using the formulation $\det(J(N_0, M_0) - \lambda I) = 0$which is referred to as the characteristic equation of the matrix $J(N_0, M_0)$. By knowing the eigenvalues, the stability around the fixed points of the system can be determined. The eigenvalues obtained for each fixed point are as follows:

1. The eigenvalues for the fixed point $E_1(0,0)$ are $\lambda_1 = a_1$ and $\lambda_2 = a_2$. The fixed point $E_1$is unstable (a node).
2. The eigenvalues for the fixed point $E_2\left(0, \frac{a_2}{b_{22}}\right)$ are $\lambda_1 = \frac{a_1 b_{22} - a_2 b_{12}}{b_{22}}$ and $\lambda_2 = \frac{(a_2 b_{22} - 2a_2 b_{22})}{b_{22}}$. There are two possible types of stability around the fixed point E_2: if (a_1b_2 - a_2b_1 < 0), then $E_2$is asymptotically stable (a node), and if (a_1b_2 - a_2b_1 > 0), then the fixed point $E_2$is unstable (a saddle point).
3. The eigenvalues for $E_3\left(\frac{(a_1 b_{22} - a_2 b_{12})}{(b_{11} b_{22} - b_{12} b_{21})}, \frac{(a_2 b_{11} - a_1 b_{21})}{(b_{11} b_{22} - b_{12} b_{21})}\right)$ are as follows:
   $\lambda_1 = \frac{-(b_{11}N_0 + b_{22}M_0) + \sqrt{(b_{11}N_0 + b_{22}M_0)^2 - 4((b_{11}b_{22} - b_{12}b_{21})N_0 M_0)}}{2}$ and
   $\lambda_2 = \frac{-(b_{11}N_0 + b_{22}M_0) - \sqrt{(b_{11}N_0 + b_{22}M_0)^2 - 4((b_{11}b_{22} - b_{12}b_{21})N_0 M_0)}}{2}$.
   There are two possible types of stability around the fixed point E_3: if ($b_{11}b_{22} - b_{12}b_{21}$ < 0), then the roots in the characteristic equation are positive and greater than $(b_{11}N_0 + b_{22}M_0)$, thus the fixed point E_3 is unstable, and if ($b_{11}b_{22} - b_{12}b_{21}$ > 0), then the roots in the characteristic equation are less than$(b_{11}N_0 + b_{22}M_0)$. Consequently, $E_3$is stable.
4. The eigenvalues for $E_4\left(\frac{a_1}{b_{11}}, 0\right)$ are $\lambda_1 = \frac{a_1 b_{11} - 2a_1 b_{11}}{b_{11}}$ and $\lambda_2 = \frac{a_2 b_{11} - a_1 b_{21}}{b_{11}}$. There are two possible types of stability around the fixed point E_4 : if $a_2 b_{11} - a_1 b_{21}$ < 0, then the fixed point is asymptotically stable (a node), and if $a_2 b_{11} - a_1 b_{21}$ > 0, then the fixed point is unstable (a saddle point).

In this article, the case study used involves two spider populations, namely Lycosa and Enoplognatha, which live together and have similarities, including similarities in food and function. In this case, the similarity is examined from the perspective of food similarity. The spider population has the highest prey similarity compared to other spider populations in the rice field habitat [7].

The parameter values in the two-population competition model with coefficients if $a_1, a_2, b_{11}, b_{12}, b_{21}$and b_{22} are positive, and the magnitude of the parameter values represents the probability of an event, with the parameter values lying within the interval [0,1]. Currently, there is no research on a specific competition model between the Lycosa population and the Enoplognatha population. To observe the influence of parameters in determining the carrying capacity of the Lycosa (*N*) and Enoplognatha (*M*) populations, it is divided into two cases with different parameter values. In first case, a stability analysis was conducted around the fixed points E_1, E_2, E_3 and E_4, with parameter values a_1 = 0.3, a_2 = 0.2, b_{11} = 0.002, b_{12} = 0.004, b_{21} = 0.001, and b_{22} = 0.003. Stability is obtained around the fixed point $E_1(0,0)$ which is unstable (a node), around the fixed point $E_2(0{,}66.7)$ which is unstable (a saddle point), around the fixed point E_3 (50,50) which is asymptotically stable (a node), and around the fixed point E_4 (150,0) which is unstable (titik pelana). This shows that the carrying capacity of the Lycosa population and the Enoplognatha population in the rice field habitat based on the two-population competition model in first case is 50 individuals each.

Orbit is a curve in the field xy that represents the solution of a two-population competition model. The solution of the model is the condition of the population size as time approaches infinity $t \to \infty$.

In this article, the orbit display of the two-population competition model was obtained with the help of Matlab software. In first case, the orbit behavior of the model was observed, where the orbit represents the condition of the Lycosa population and the Enoplognatha population with parameter a_1 = 0.3, a_2 = 0.2, b_{11} = 0.002, b_{12} = 0.004, b_{21} = 0.001, and b_{22} = 0.003. Below is the image of the orbit behavior around the fixed point of the two-population competition model in first case .

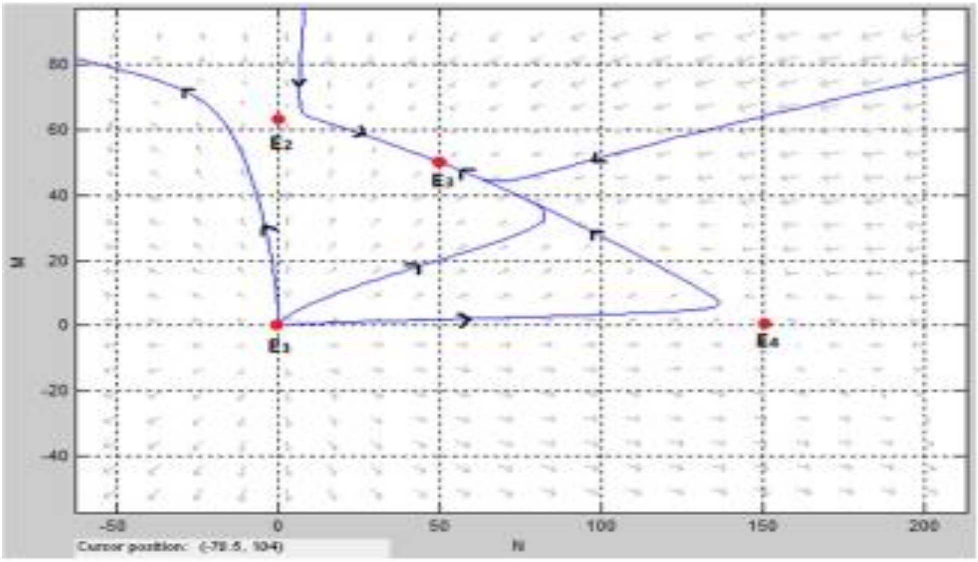

Figure 2. Orbit around the fixed point of the two-population competition model

In Figure 2, the orbital behavior of the two-population competition model can be observed. In the picture, it can be seen that the orbits of the Lycosa and Enoplognatha population numbers, which move away from the fixed point E_1, move closer to the fixed point E_2. However, after that, the orbits of the two spider populations, which initially approached the fixed point E_2, then moved away from it. This is because the stability around the fixed point E_2 is unstable (a saddle point). Then the orbits that move away from the fixed points E_1 and E_2 approach the fixed points $E_3$and E_4. The orbits of the Lycosa population and the Enoplognatha population that approach the fixed point E_3 move towards that fixed point. This is because the stability around the fixed point E_3 is stable (node point). Meanwhile, the orbit that initially moves towards the fixed point E_4 then moves away from that fixed point. This is because the stability around the fixed point E_4 is unstable (a saddle point).

From the simulation results, to observe the stability around the equilibrium point of the two-population competition model, one can look at the behavior of the orbit around that equilibrium point. In case I, out of the 4 equilibrium points, only one is stable, which is E_3 (50,50). The orbit around E_3 (50,50) moves towards that point. The aforementioned point represents a condition where both populations coexist. This indicates that with parameter values $_1 = 0.3$, $a_2 = 0.2$, $b_{11} = 0.002$, $b_{12} = 0.004$, $b_{21} = 0.001$, and $b_{22} = 0.003$, the Lycosa and Enoplognatha populations will be in a stable state with each having a number of 50 individuals. From the analysis, the carrying capacity for the Lycosa and Enoplognatha populations in the rice field habitat based on the two-population competition model is each 50 individuals. Therefore, no matter how great the disturbance received by the rice field habitat, the population of the two spider species will change without deviating from or exceeding the system's carrying capacity for a long time. Thus, even if there is partial overlap between the two spider populations, it does not cause either or both populations to become extinct. In other words, the carrying capacity maintains the balance in the rice field habitat, allowing the two spider populations to coexist for a long time.

In second case , the orbital behavior of the model was also observed, where the orbit represents the condition of the Lycosa population and the Enoplognatha population but with different parameter values compared to case I, specifically with values $a_1 = 0.2$, $a_2 = 0.3$, $b_{11} = 0.004$, $b_{12} = 0.002$, $b_{21} = 0.001$, and $b_{22} = 0.008$. Below is the image of the orbital behavior around the fixed point of the two-population competition model in second case .

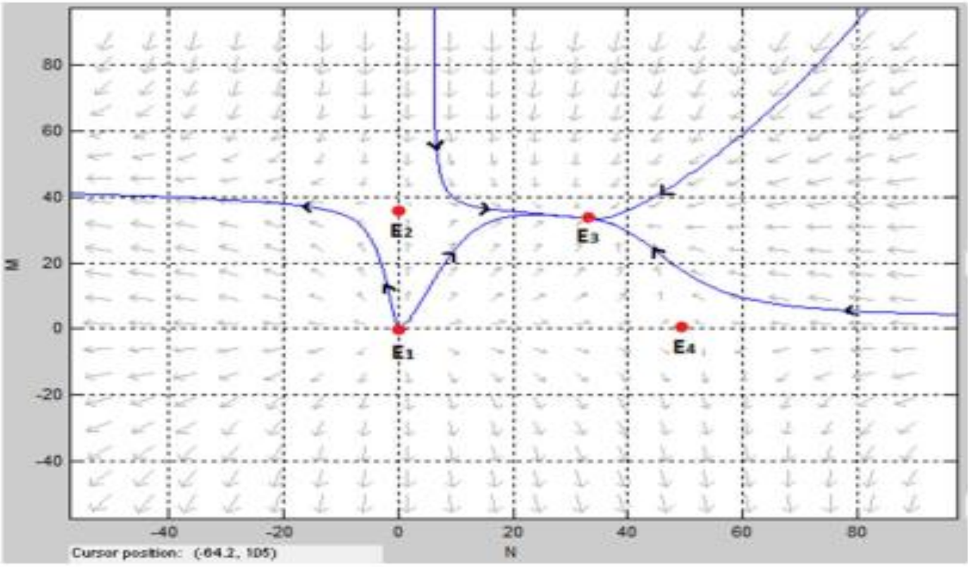

Figure 3. Orbit around the fixed point of the two-population competition model

In Figure 3, it can be seen that the orbit of the Lycosa population and the Enoplagnatha population, which initially moved away from the fixed point E_1, moved closer to the fixed point E_2. However, after that, the orbit of the two spider populations, which initially moved closer to the fixed point E_2, moved away from their fixed point. Similarly to first case , this is because the stability around the fixed point E_2 is unstable (saddle point). Then, the orbits that move away from the fixed points E_1 and E_2 approach the fixed points E_3 and E_4. The orbit that approaches the fixed point E_3 moves towards that fixed point. This is because the stability around the fixed point E_3 is stable (node point). Meanwhile, the orbit that initially moves towards the fixed point E_4 then moves away from that fixed point. This is because the stability around the fixed point E_4 is unstable.

In second case , there is also only one stable fixed point, namely E_3 (33.3, 33.3). The orbits of the populations of Lycosa and Enoplagnatha spiders around E_3 (33.3, 33.3) move towards that point. This indicates that with parameter values values $_1$ = 0.2, a_2 = 0.3, b_{11} = 0.004, b_{12} = 0.002, b_{21} = 0.001, and b_{22} = 0.008 the Lycosa population and the Enoplagnatha population will be in a stable state with a number of 33.3 individuals. From the analysis results, the carrying capacity of the system for the first population is 33.3 individuals and for the second population is 33.3 individuals. As in first Case , regardless of the extent of the disturbance experienced by the rice field habitat, the number of both populations will change only around or more than the value of 33.3 individuals in a short time. By knowing the carrying capacity of the rice field habitat, the balance of the habitat can be maintained.

Conclusion

Based on the results of the analysis and discussion, the following conclusions can be drawn:

1. The fixed points of the two-population competition model obtained are as follows:
   a. $E_1(0,0)$
   b. $E_2\left(0,\frac{a_2}{b_{22}}\right)$
   c. $E_3\left(\frac{(a_1b_{22}-a_2b_{12})}{(b_{11}b_{22}-b_{12}b_{21})},\frac{(a_2b_{11}-a_1b_{21})}{(b_{11}b_{22}-b_{12}b_{21})}\right)$
   d. $E_4\left(\frac{a_1}{b_{11}},0\right)$

2. Based on case I and case II, the stability around the fixed points of the two-population competition model is as follows:

a. The fixed point $E_1(0,0)$ is unstable.

b. The fixed point $E_2\left(0,\frac{a_2}{b_{22}}\right)$ is unstable.

c. The fixed point$E_3\left(\frac{(a_1b_{22}-a_2b_{12})}{(b_{11}b_{22}-b_{12}b_{21})},\frac{(a_2b_{11}-a_1b_{21})}{(b_{11}b_{22}-b_{12}b_{21})}\right)$ is stable.

a. d. The fixed point $E_4\left(\frac{a_1}{b_{11}},0\right)$ is unstable.

3. The behavior of the orbit around the fixed points of the two-population competition model will move away from the unstable fixed points and approach the stable fixed points. Therefore, to determine the stability of the fixed points, it is not only seen from the eigenvalue (λ), but also from the behavior of the orbit around those fixed points.

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